Perturbative nature of color superconductivity

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Color superconductivity is a possible phase of high density QCD. We present a systematic derivation of the transition temperature T_C from the QCD Lagrangian through study of the di-quark proper vertex. With this approach, we confirm the dependence of T_C on the coupling g, namely $T_C \sim \mu g^{-5} e^{-\kappa/g}$, previously obtained from the one-gluon exchange approximation in the superconducting phase. The diagrammatic approach we employ allows us to examine the perturbative expansion of the vertex and the propagators. We find an additional $O(1)$ contribution to the prefactor of the exponential from the one-loop quark self energy and that the other one-loop radiative contributions and the two gluon exchange vertex contribution are subleading.

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I. INTRODUCTION

Color superconductivity is a possible phase of high density QCD, pioneered by Bailin and Love and others [1], who pointed out that the $\bar{3}$ channel of a di-quark interaction is attractive through one gluon exchange. Recently, using an effective four-fermion interaction in the superconducting phase, much work has been carried out on examining an energetically favored condensate which, for $N_f = 3$, breaks the original $SU(3)^c \times SU(3)^{f_L} \times SU(3)^{f_R}$ symmetry to its diagonal symmetry [2–5]. This color superconductivity mechanism has been called color-flavor locking.

Several attempts have been made to determine the parameters of the superphase from the QCD action [6–12]. In all cases one arrives at the same dependence of the zero-temperature gap energy on the QCD running coupling constant evaluated at the Fermi energy, namely

$$\Delta \sim \mu g^{-5} e^{-\kappa/g}, \qquad (1.1)$$

where μ is the chemical potential and g the running Yang-Mills coupling evaluated at μ. This dependence upon the coupling differs significantly from the BCS case, $\Delta \sim \mu e^{-\kappa'/g^2}$, due to the long range propagation of magnetic gluons. The coefficient $\kappa = 3\pi^2/\sqrt{2}$ was first derived by Son [7], and subsequently verified in [9] and [10,11]. The latter two of these generalized this scaling behavior to the transition temperature and in addition derived its ratio to the zero temperature gap. All of these results were obtained from superconducting gap equations with one gluon exchange. As such, although they contain the correct leading order behavior that determines the dependence of the gap on the coupling, more detailed calculations are required to recover all the leading order contributions to the pre-exponential factor.

In contrast with previous works, we approach the transition temperature from the normal phase, with $T \ll \mu$. There are several advantages with this approach. First of all, the propagators in the normal phase are not subjected to modifications from the long range order, the form of which are ansatz dependent. This ensures that issues of gauge invariance and higher order corrections are relatively simple to handle. Secondly, the integral equation for the proper vertex function, which determines the pairing instability, is linear while the gap equation in the superphase is nonlinear. Thirdly, the hard dense loop contribution to the gluon propagator is free from the Meissner effect. Though it has not been taken into account so far in the gap equation calculations in the superphase, the Meissner effect is nevertheless expected to alter the pre-exponential factor [9].

Working directly from the QCD Lagrangian provides a natural framework within which to examine the perturbative nature of the theory at high density. Indeed, we shall find that interactions of second order, $O(g^4)$, make a leading order contribution to the pre-exponential factor and that all higher order contributions are subleading. The sum of these effects suggests that a derivation of the transition temperature from the normal phase will not only provide a rigorous verification of the results obtained from within the superphase, but should also allow a clean and exact determination of the pre-exponential factor. Combined with the aesthetic benefit of calculating directly from the QCD Lagrangian, this more than compensates for the technical complexity of this approach.

Starting from the $SU(N)$ QCD Lagrangian, the calculation of the transition temperature can be cast easily into thermal diagrams with gauge invariance manifest. Following the formulation developed by Gor'kov and Melik-Barkhudarov for the non-relativistic Fermi-gas [13], which allows for a
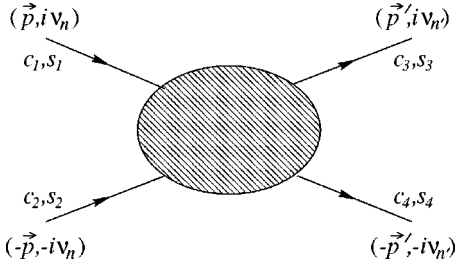
FIG. 1. Proper vertex function, $\Gamma^{c_3,c_4;c_1,c_2}_{s_3,s_4;s_1,s_2}(n',n|\vec{p}',\vec{p})$.

systematic examination of the higher order contributions, we calculate the transition temperature to leading order in coupling and obtain

$$\pi k_B T_C = \mu g^{-5} c c' e^{-\sqrt{(6N)/(N+1)}(\pi^2/g)}, \quad (1.2)$$

where up to a constant of $O(1)$, $c = 1024\sqrt{2}\,\pi^4 N_f^{-5/2}$, in agreement with [9–11], and the exponent, $-\sqrt{6N/(N+1)}\,\pi^2/g$, in agreement with [7,9–11], are determined by the leading order one gluon exchange process. The previously unreported factor

$$c' = \exp\left[-\frac{1}{16}(\pi^2+4)(N-1)\right] \simeq 0.1766 \quad \text{for } N=3, \quad (1.3)$$

comes from the logarithmic suppression of the quasi-particle weight in the dressed quark propagator. For Landau damping, obtained in the hard dense loop approximation, the contribution to the prefactor from two gluon exchange diagrams is subleading in g. This, however, is not the case for a hypothetical static screening case where the perturbative nature is completely spoiled by infrared log-enhancement in higher orders.

II. CALCULATION OF THE QCD TRANSITION TEMPERATURE

We consider an $SU(N)$ color gauge field coupled to N_f flavors of massless quarks with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \bar{\psi}_f\gamma_\mu(\partial_\mu - igA_\mu)\psi_f, \quad (2.1)$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, $A_\mu = A^a_\mu t^a$ with t^a the $SU(N)$ generator in its fundamental representation. Since the Lagrangian (2.1) is diagonal with respect to both flavor and chirality, the corresponding indices will be dropped below.

We derive the transition temperature by investigating the onset of the pairing instability in the proper vertex function corresponding to the scattering of two quarks at non-zero temperature and chemical potential. This vertex function with zero total momentum and zero total Matsubara energy, $\Gamma^{c_3,c_4;c_1,c_2}_{s_3,s_4;s_1,s_2}(n',n|\vec{p}',\vec{p})$, is shown in Fig. 1, where n and n' label the Matsubara energies $i\nu_n = (2\pi i/\beta)(n+\frac{1}{2})$ of individual quarks. Each of the superscripts c, which denote color, are associated with a leg. The subscripts s, which label
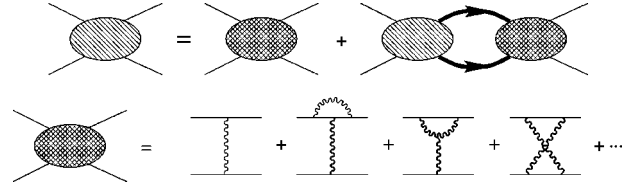
FIG. 2. The Schwinger-Dyson equation. As in Fig. 1, Γ is represented by singly hashed vertices and $\bar{\Gamma}$ is represented by doubly hashed vertices. The expansion of $\bar{\Gamma}$ is given up to $O(g^4)$.

the states above or below the Dirac sea, are either $+$ or $-$. We find it convenient for the partial wave analysis to associate the Dirac spinors $u(\vec{p})$ and $v(\vec{p})$, which satisfy the Dirac equations $(\gamma_4 p - i\vec{\gamma}\cdot\vec{p})u(\vec{p})=0$ and $(\gamma_4 p - i\vec{\gamma}\cdot\vec{p})v(\vec{p})=0$, to the quark-gluon vertex instead of to the quark propagator. Therefore, where $s=(s_1,s_2)$ represents the incoming subscripts and $s'=(s_3,s_4)$ represents the outgoing subscripts, suppressing the color indices and momentum-energy dependence, we write

$$\Gamma_{s';s} = \bar{U}_\gamma(s_3,\vec{p}')\bar{U}_\delta(s_4,-\vec{p}')\Gamma_{\gamma\delta,\alpha\beta}U_\alpha(s_1,\vec{p})U_\beta(s_2,-\vec{p}). \quad (2.2)$$

The vertex function $\Gamma_{\gamma\delta,\alpha\beta}$ is given by conventional Feynman rules, and $U(s,\vec{p})=u(\vec{p})$ or $v(-\vec{p})$ for $s=+$ or $-$, respectively. The proper vertex function satisfies a Dyson-Schwinger equation as shown in Fig. 2. This integral equation of Fredholm type may be written with all indices suppressed as

$$\Gamma(n',n|\vec{p}',\vec{p}) = \bar{\Gamma}(n',n|\vec{p}',\vec{p}) + \frac{1}{\beta}\sum_m \int \frac{d^3\vec{q}}{(2\pi)^3}$$
$$\times K(n',m|\vec{p}',\vec{q})\Gamma(m,n|\vec{q},\vec{p}), \quad (2.3)$$

where $\bar{\Gamma}$ represents the two quark irreducible vertex with all indices defined in the same way as for Γ. The kernel has the explicit form

$$K^{c_3,c_4;c_1,c_2}_{s_3,s_4;s_1,s_2}(n',m|\vec{p}',\vec{q}) = \bar{\Gamma}^{c_3,c_4;c_1,c_2}_{s_3,s_4;s_1,s_2}(n',m|\vec{p}',\vec{q})S_{s_1}(m|\vec{q})$$
$$\times S_{s_2}(-m|-\vec{q}), \quad (2.4)$$

where we write $S_s(n|\vec{p})$ for the full quark propagator with momentum $\vec{p}$ and Matsubara energy $i\nu_n$. The zeroth order quark propagator reads

$$S_s(n|\vec{p}) = \frac{i}{i\nu_n - sp + \mu}, \quad (2.5)$$

and the diagrammatic expansion of $\bar{\Gamma}$ to $O(g^4)$ is displayed in Fig. 2.

The transition temperature may be obtained from Eq. (2.3) by examining the Fredholm determinant, $\mathcal{D} \equiv \det(1 - K)$, which is a function of T and μ. T_C is then given by the highest temperature at which $\mathcal{D}(T,\mu)$ vanishes. To demonstrate the gauge invariance of this formulation, we need to
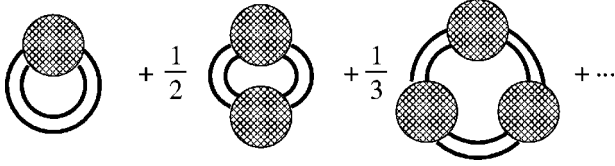
FIG. 3. The bubble diagrams.

extend the integral equation (2.3) to include scattering with arbitrary total momentum and energy. Denote by K' the kernel for which either the total momentum or the total energy or both are nonzero. Then the Fredholm determinant with arbitrary total momentum and energy factorizes as

$$\hat{\mathcal{D}} = \det(1-K)\det(1-K'). \qquad (2.6)$$

On the other hand, $\ln \hat{\mathcal{D}}$ is given by the sum of bubble diagrams shown in Fig. 3. They are in fact manifestly gauge invariant as may be seen following the argument given in [14].

For the rest of this paper, we shall work in Coulomb gauge, in which the full gluon propagator takes the form

$$D_{ij}(\vec{k},\omega) = D^M(\vec{k},\omega)\left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2}\right), \qquad (2.7)$$

$$D_{44}(\vec{k},\omega) = D^E(\vec{k},\omega), \qquad (2.8)$$

and

$$D_{4j}(\vec{k},\omega) = D_{j4}(\vec{k},\omega) = 0. \qquad (2.9)$$

Since Γ corresponds to di-quark scattering it can be decomposed into irreducible representations of $SU(N)$ by either symmetrization [representation **6** for $SU(3)$] or antisymmetrization [representation $\bar{\mathbf{3}}$ for $SU(3)$] among the initial and final color indices, i.e.,

$$\Gamma^{c',c}_{s',s}(n',n|\vec{p}',\vec{p}) = \sqrt{2}\,\delta^{c_1(c_3}\delta^{c_4)c_2}\Gamma^S_{s',s}(n',n|\vec{p}',\vec{p})$$
$$+ \sqrt{2}\,\delta^{c_1[c_3}\delta^{c_4]c_2}\Gamma^A_{s',s}(n',n|\vec{p}',\vec{p}), \qquad (2.10)$$

where $(\cdots)$ and $[\cdots]$ denote symmetrization and antisymmetrization with weight one, respectively. $\tilde{\Gamma}$ may be decomposed similarly. Since the Fermi surface has a pairing instability in the presence of even an arbitrarily weak attractive interaction, as is also the case for BCS theory, we need only focus on the attractive antisymmetric channel for non-Abelian theories.

Both $\tilde{\Gamma}^A_{s',s}(n',n|\vec{p}',\vec{p})$ and $\Gamma^A_{s',s}(n',n|\vec{p}',\vec{p})$ can be expanded in terms of Legendre polynomials, i.e.,

$$\Gamma^A_{s',s}(n',n|\vec{p}',\vec{p}) = \sum_l \gamma^l_{s',s}(n',n|p',p)P_l(\cos\theta). \qquad (2.11)$$

Substituting such expansions into Eq. (2.3), we find another Fredholm equation satisfied by $\gamma^l_{s',s}(n',n|p',p)$:

$$\gamma^l_{s',s}(n',n|p',p) = \tilde{\gamma}^l_{s',s}(n',n|p',p) + \frac{1}{\beta}\sum_{m,s''}\int_0^\infty dq$$
$$\times K^l_{s',s''}(n',m|p',q)\gamma^l_{s'',s}(m,n|q,p), \qquad (2.12)$$

where the kernel $K^l_{s',s}$ has the form

$$K^l_{s',s}(n',n|p',p) = \frac{p^2\,\tilde{\gamma}^l_{s',s}(n',n|p',p)}{2\pi^2(2l+1)}S_{s_1}(n|p)S_{s_2}(-n|p). \qquad (2.13)$$

We consider the $l=0$ term in the partial wave expansion, although the higher partial wave terms may contribute [9].

The Fredholm determinant of Eq. (2.12) with $l=0$ can be written as $\mathcal{D} = \Pi_j(1-\lambda_j^{-2})$ with λ_j^2 the eigenvalues (labeled by the integer suffix j) defined by the homogeneous equation,

$$f_{s'}(n,p) = \lambda^2\frac{1}{\beta}\sum_{m,s}\int_0^\infty dq\, K^0_{s',s}(n,m|p,q)f_s(m,q). \qquad (2.14)$$

At sufficiently high temperature, all $\lambda_j^2 > 1$, so that $\mathcal{D}\neq 0$ and there is no instability—the theory is in the normal phase. As the temperature is reduced, we find the transition temperature to the superconducting phase is that at which the smallest of $\{\lambda_j^2\}$ reaches one. The solution of Eq. (2.14) provides the eigenvalues in terms of the parameters of the theory; the temperature, coupling and chemical potential. Hence the inversion of $\lambda_0^2(T,g,\mu)=1$, where λ_0 is the smallest eigenvalue, yields the transition temperature T_c.

In the presence of a Fermi sea, hard dense loops have to be included in the gluon propagator. As a result, the Coulomb interaction is strongly screened by the Debye length, $\lambda_D = m_D^{-1}$, where

$$m_D^2 = \frac{N_f g^2}{\pi^2}\int_0^\infty dq\, q\frac{1}{e^{\beta(q-\mu)}+1} \simeq \frac{N_f g^2\mu^2}{2\pi^2}. \qquad (2.15)$$

The dressed Coulomb propagator at momentum-energy $(\vec{k},i\omega)$ reads

$$D^E(\vec{k},\omega) = \frac{-i}{\vec{k}^2+\sigma^E(\vec{k},\omega)}, \qquad (2.16)$$

with $\sigma^E(\vec{k},\omega)\simeq m_D^2$ for $\omega \ll k \ll \mu$. However, the magnetic interaction is poorly screened. While a magnetic mass may exist, of order T, Landau damping [15] prevails at $\mu \gg k_B T$. In this case the propagator for a magnetic gluon is

$$D^M(\vec{k},\omega) = \frac{-i}{\vec{k}^2+\omega^2+\sigma^M(\vec{k},\omega)}. \qquad (2.17)$$

The only region of infrared sensitivity on the (k,ω)-plane is $\omega \ll k \ll \mu$ where $\sigma^M(\vec{k},\omega) \simeq (\pi/4) m_D^2 (|\omega|/|\vec{k}|)$. To g^2 order, the contribution to $\tilde{\gamma}_{++}^0(n',n|p',p)$ arises from the one-gluon exchange diagram of Fig. 2 and is given by

$$\tilde{\gamma}_{++}^0(n',n|p',p) = -\frac{g^2}{12p'p}\left(1+\frac{1}{N}\right)$$
$$\times \left[\ln\frac{8\mu^3}{|p'-p|^3 + \frac{\pi}{4}m_D^2|\nu_{n'}-\nu_n|} + \frac{3}{2}\ln\frac{4\mu^2}{m_D^2} \right], \quad (2.18)$$

where a term finite in the limit $g \to 0$ has been dropped. To the leading order of $\ln \mu/k_B T$, the summation over Matsubara energy can be replaced by an integral and $|p'-p|^3$ ignored. Furthermore, the components f_{+-}, f_{-+}, and f_{--} can be neglected and the integral over q can be carried out for a solution smooth in the neighborhood of the Fermi sea, $p \simeq \mu$. Equation (2.14) is then approximated by

$$f(\nu) = \int_\epsilon^{\nu_0} \frac{d\hat{\nu}'}{\hat{\nu}'} \mathcal{K}(\nu,\nu') f(\nu'), \quad (2.19)$$

where

$$\hat{\nu} = \frac{N_f^{5/2} g^5}{1024\sqrt{2}\pi^4\mu}\nu, \quad (2.20)$$

and

$$\epsilon = \frac{N_f^{5/2} g^5 k_B T}{1024\sqrt{2}\pi^3\mu}. \quad (2.21)$$

The reduced kernel $\mathcal{K}(\nu,\nu')$ is given by

$$\mathcal{K}(\nu,\nu') = \frac{\lambda^2}{2}\frac{g^2}{24\pi^2}\left(1+\frac{1}{N}\right)\left(\ln\frac{1}{|\hat{\nu}-\hat{\nu}'|} + \ln\frac{1}{|\hat{\nu}+\hat{\nu}'|}\right), \quad (2.22)$$

with an ultraviolet cutoff $\nu_0 \sim 1$ introduced. The eigenvalue problem (2.19) can be solved using the same approximation employed by Son [7], which amounts to replacing the kernel of Eq. (2.19) with $(2/\hat{\nu}')\ln 1/\hat{\nu}_>$ where $\hat{\nu}_> = \max(\hat{\nu},\hat{\nu}')$. We find that

$$\lambda_j^2 \frac{g^2}{24\pi^2}\left(1+\frac{1}{N}\right)\ln^2\frac{1}{\epsilon} = \left(j+\frac{1}{2}\right)^2\pi^2, \quad (2.23)$$

where j is an integer and

$$f(\nu) \simeq \sqrt{\frac{2}{\ln 1/\epsilon}}\sin\left[\left(j+\frac{1}{2}\right)\pi\frac{\ln 1/\hat{\nu}}{\ln 1/\epsilon}\right]. \quad (2.24)$$

Setting the smallest eigenvalue to one, namely $\lambda_0^2 = 1$, we finally arrive at the exponent and the prefactor c of Eq. (1.2).

Here we wish to highlight the mathematical structure of Eq. (2.3) which characterizes the long range attractive interaction. If, instead, the pairing interaction was of a short range nature, the transition temperature could be located by means of the standard expansion of the Fredholm determinant

$$\mathcal{D}(\lambda) = 1 - \frac{1}{\beta}\sum_n \int_0^\infty dp\, K(n,p|n,p) + \frac{1}{2\beta^2}\sum_{n,n'}\int_0^\infty dp \int_0^\infty dq \begin{vmatrix} K(n,p|n,p) & K(n,p|n',q) \\ K(n',q|n,p) & K(n',q|n',q) \end{vmatrix} + \cdots. \quad (2.25)$$

The mth term in the expansion on the right hand side contains m kernels folded together and for a short range interaction would be of the order $g^{2m}\ln(1/\epsilon)$ since there is only one eigenvalue of K which diverges as $\ln(1/\epsilon)$ in the limit $\epsilon \to 0$. At the transition temperature one would have $g^2\ln(1/\epsilon) \sim 1$, and thus the order of magnitude of the subsequent terms would be $g^{2(m-1)}$. Therefore only the first two non-trivial terms ($m=1,2$) would be sufficient to determine the transition temperature up to the leading order of the pre-exponential factor. On the other hand, for the present long range attraction of the QCD model, the logarithm in the kernel of Eq. (2.14) makes the mth term of the expansion (2.25) of the order $g^{2m}\ln^{2m}(1/\epsilon)$ since there are now an infinite number of eigenvalues of K which diverge as $\ln^2(1/\epsilon)$ in the limit $\epsilon \to 0$, as indicated in Eq. (2.23). Hence the series cannot be truncated at T_C, and a new method has to be employed to fix both the exponent and the prefactor.

Note that the second term on the right hand side of Eq. (2.25) is in fact logarithmically infinite for the kernel (2.13). On the other hand, this term corresponds to the sum $\Sigma\lambda_j^{-2}$ which is convergent according to Eq. (2.23). The reason for this apparent paradox lies in Son's approximation, which led to the eigenvalues given in Eq. (2.23). When the approximation is corrected, the eigenvalues instead become

$$\lambda_j^{-2} = \frac{g^2}{24\pi^4}\left(1+\frac{1}{N}\right)\left[\left(j+\frac{1}{2}\right)^{-2}\ln^2\frac{1}{\epsilon} + c_j\right], \quad (2.26)$$

with $c_j \sim j^{-1}$ for $j \gg 1$ [16], which explains the appearance of the logarithmic divergence in this term when summing over j.

It is interesting to note that a similar non-BCS scaling behavior of the type indicated in Eq. (1.1) was obtained via a

mean field calculation of 2D superconductivity at the von Hove singularity of the electronic density of states [17].

III. HIGHER ORDER CORRECTIONS

We now come to the question of higher order corrections to the kernel from the perturbative expansion of the quark propagator and the irreducible vertex $\bar{\Gamma}$. These become important if sufficient powers of the infrared logarithm accompany the coupling constant g^2. It follows from Eq. (1.2) that $\ln \mu/k_B T_C \sim 1/g$ at the transition temperature. Thus, if the $O(g^4)$ contribution to $K^0_{s',s}$ is of the form $g^4 \ln^\delta \mu/|\nu_{n'} - \nu_n|$ or $g^4 \ln^\delta \mu/|\nu_{n'}|$, its magnitude relative to the $O(g^2)$ term given by Eq. (2.18) will be $g^{3-\delta}$. The perturbative robustness of the exponent then requires that $\delta < 3$; robustness of the prefactor requires $\delta < 2$. We find with Landau damping that $\delta = 2$; one logarithm originates from the leading order of $\tilde{\gamma}$, Eq. (2.18), and the second from the self-energy of quarks [7]. Therefore there will be an $O(1)$ correction to the prefactor.

Parametrizing the dressed quark propagator above the Dirac sea as

$$S_+(p_0, \vec{p}) = \frac{i}{p_0 - p + \mu - \Sigma(p_0, \vec{p})}, \qquad (3.1)$$

we find, to one loop order, that

$$\frac{\partial}{\partial p_0} \text{Re} \, \Sigma(p_0, \vec{p})\big|_{p=\mu} = -\frac{N^2-1}{N} \frac{g^2}{24\pi^2} \ln \frac{\mu^3}{m_D^2 \max(|p_0|, k_B T)}, \qquad (3.2)$$

while $(\partial/\partial p) \text{Re} \, \Sigma(p_0, \vec{p})\big|_{p=\mu}$ remains finite in the limit $p_0 \to 0$. For a Matsubara energy $p_0 = i\nu_n = i(2\pi/\beta)(n + \frac{1}{2})$, we thus have

$$S_+(n|p) = \frac{i}{i\left(1 + \dfrac{N^2-1}{N} \dfrac{g^2}{24\pi^2} \ln \dfrac{\mu^3}{m_D^2 |\nu_n|}\right)\nu_n - p + \mu}. \qquad (3.3)$$

Following the steps which lead from Eq. (2.16) to Eq. (2.22) above, the inclusion of this effect amounts to replacing the reduced kernel $\mathcal{K}(\nu, \nu')$ by $\mathcal{K}(\nu, \nu') + \Delta\mathcal{K}(\nu, \nu')$ where

$$\Delta\mathcal{K}(\nu, \nu') = -\frac{N^2-1}{N} \frac{g^2}{24\pi^2} \mathcal{K}(\nu, \nu') \ln \frac{1}{\hat{\nu}'}. \qquad (3.4)$$

Treating $\Delta\mathcal{K}(\nu, \nu')$ as a perturbation, the shift of the eigenvalue in Eq. (2.23) with $j=0$ turns out to be

$$\delta \frac{1}{\lambda_0^2} = \int_\epsilon^{\nu_0} \frac{d\hat{\nu}}{\hat{\nu}} \int_\epsilon^{\nu_0} \frac{d\hat{\nu}'}{\hat{\nu}'} f_0(\nu) \Delta\mathcal{K}(\nu, \nu') f_0(\nu')$$

$$= -\frac{2(\pi^2+4)}{\pi^4}\left(1 + \frac{1}{N}\right)\frac{N^2-1}{N}\left(\frac{g^2}{24\pi^2}\right)^2 \ln^3 \frac{1}{\epsilon}. \qquad (3.5)$$

The condition for the critical temperature, $\lambda_0^2 = 1$, now becomes

$$\frac{g^2}{6\pi^4}\left(1 + \frac{1}{N}\right)\ln^2 \frac{1}{\epsilon}$$

$$-\frac{2}{\pi^4}\left(\frac{g^2}{24\pi^2}\right)^2\left(1 + \frac{1}{N}\right)\frac{N^2-1}{N}(\pi^2+4)\ln^3 \frac{1}{\epsilon} = 1, \quad (3.6)$$

the solution of which, for small g, gives rise to the result (1.2) with both c and c'. The logarithmic dependence of Eq. (3.2) upon the coupling constant g, written in m_D^2, will change the prefactor g^{-5} of Eq. (1.2) to $g^{-5+O(g)}$. However this correction is of higher order.

Other higher order corrections to $\tilde{\gamma}$ have also been partially addressed in the literature. The vertex correction has been discussed in [9] and some renormalization group arguments have been applied to the straight box diagrams of two gluon exchange [7], which was conjectured to be of the same order as the crossed box diagram. For the case of Landau damping, our analysis of the vertex correction is in agreement with [9], indicating an $O(g)$ contribution to the prefactor. We also find that the crossed box diagram is free from any logarithmic enhancement and so its contribution is suppressed relative to the one gluon exchange by a factor of $g^2/\ln(\mu/k_B T)$ [18]. On the other hand, the straight box diagram, which corresponds to the convolution of two single gluon exchange kernels, is logarithmically divergent at $T = 0$ and contains all the powers of logarithms at $T \neq 0$ necessary to produce the result of the ladder sum implicit in Eq. (2.14).

The crossed box contributes to $\tilde{\gamma}^0_{++}$ a term

$$B = -\frac{1}{2}g^4 \int_{-1}^1 d\cos\theta \int_{-\infty}^\infty \frac{d\omega}{2\pi} \int \frac{d^3\vec{l}}{(2\pi)^3} D_{\mu\mu'}\left(\vec{l} - \frac{\vec{q}}{2}, i\omega\right) D_{\nu\nu'}\left(\vec{l} + \frac{\vec{q}}{2}, i\omega\right) [\bar{u}(\vec{p}')\gamma_\mu S_F(\vec{P} + \vec{l}, i\omega)\gamma_\nu u(\vec{p})]$$

$$\times [\bar{u}(-\vec{p}')\gamma_{\nu'} S_F(-\vec{P} + \vec{l}, i\omega)\gamma_{\mu'} u(-\vec{p})], \qquad (3.7)$$

where $|\vec{p}| = |\vec{p}'| = \mu$, $\vec{P} = \frac{1}{2}(\vec{p} + \vec{p}')$, $\vec{q} = \vec{p} - \vec{p}'$ and the summation over Matsubara energy has been replaced by an integral for $T \ll \mu$. Ignoring the Coulomb propagator, the contribution from the small scattering angle, $|\theta| < \theta_0 \ll 1$, and infrared region, IR: $|\vec{l}| \ll \mu$ and $|\omega| \ll \mu$,

$$B_{IR} = -\frac{1}{2}g^4 \int_{-\theta_0}^{\theta_0} d\theta \sin\theta \int_{IR} \frac{d\omega}{2\pi} \frac{d^3\vec{l}}{(2\pi)^3} D_{ii'}\left(\vec{l}-\frac{\vec{q}}{2},i\omega\right) D_{jj'}\left(\vec{l}+\frac{\vec{q}}{2},i\omega\right) [\bar{u}(\vec{p}')\gamma_i S_F(\vec{P}+\vec{l},i\omega)\gamma_j u(\vec{p})]$$

$$\times [\bar{u}(-\vec{p}')\gamma_{j'} S_F(-\vec{P}+\vec{l},i\omega)\gamma_{i'} u(-\vec{p})], \tag{3.8}$$

is bounded: $B_{IR} \leq b$ where

$$b \equiv \frac{1}{32\pi^4\mu^2} \int_0^{\theta_0} d\theta \int_{IR} d\rho \, d^3\vec{r} \frac{r_+ r_- |E_+ E_- - \rho^2|}{(r_+^3+\kappa|\rho|)(r_-^3+\kappa|\rho|)(\rho^2+E_+^2)(\rho^2+E_-^2)}. \tag{3.9}$$

Here $E_\pm = |\vec{P}\pm\vec{l}|/\mu - 1$, $r_\pm = |\vec{l}\pm\vec{q}|/\mu$, $\rho=|\omega|/\mu$ and $\kappa=(\pi/4)(m_D^2/\mu^2)$. Transforming the integration variables from $\theta,\vec{r}$ to $E_\pm, r_\pm$, we end up with $b=(1/32\pi^4\mu^2)\int_0 d\rho K(\rho)$ where

$$K(\rho) = \int dE_+ dE_- dr_+^2 dr_-^2 \, J \frac{r_+ r_- |E_+ E_- - \rho^2|}{(r_+^3+\kappa|\rho|)(r_-^3+\kappa|\rho|)(\rho^2+E_+^2)(\rho^2+E_-^2)}, \tag{3.10}$$

with the Jacobian

$$J = [(E_+ - E_-)^4 - 4(r_+^2 + r_-^2)(E_+ - E_-)^2$$
$$- 16(E_+ + E_-)^2 + 16r_+^2 r_-^2]^{-1/2}. \tag{3.11}$$

As $\rho \to 0$, we find that $K(\rho) \to \text{const} \times \rho^{-2/3}$ up to some power of $\ln\rho$. Therefore B_{IR} as well as B is free from infrared divergences.

For the sake of comparison, we have also examined the $O(g^4)$ corrections with only a static mass gap for gluons, $m \ll \mu$. This amounts to replacing the magnetic gluon propagator (2.17) by

$$D^M(\vec{k},\omega) = \frac{-i}{\omega^2 + \vec{k}^2 + m^2}. \tag{3.12}$$

The integration over the Euclidean energy can be carried out readily by the residue theorem and the remaining integral can be classified according to the contributions from the gluon and quark poles. Denoting the $O(g^4)$ contribution to $\tilde{\gamma}_{++}^0$ by $\Delta\tilde{\gamma}$, we find to the leading order in $\ln\mu/m$

TABLE I. Leading log coefficients.

| | Transverse | | Covariant | |
	with FS	without FS	with FS	without FS
g_1	$-1/4$	$-1/4$	0	0
q_1	0	0	0	0
g_2	0	0	$-2/3$	$-2/3$
q_2	$1/16$	0	1	0

$$\Delta\tilde{\gamma} = \frac{g^4}{2\pi^2}\left[\frac{1}{2N}\left(1+\frac{1}{N}\right)c_1 - \frac{1}{4}\left(N-\frac{2}{N}-\frac{1}{N^2}\right)c_2\right]\ln^3\frac{\mu}{m}, \tag{3.13}$$

where c_1 corresponds to the vertex corrections and c_2 corresponds to the crossed box diagram in Fig. 2. We write $c_1 = g_1 + q_1$ and $c_2 = g_2 + q_2$, where g_1 and g_2 come from the gluon poles and q_1 and q_2 come from the quark poles, respectively. The g and q coefficients are tabulated in Table I for various cases. This presence of $\ln^3\mu/m$ will ruin the perturbative nature of the formulation, provided $m \sim T$. At this point, the difference between Landau damping and static screening is clear. To the leading order of one gluon exchange, the difference merely amounts to the substitution of the static screening mass by $m_D^{2/3}|\omega|^{1/3}$, with ω the Euclidean energy transfer and $|\omega| \ll \mu$. This is not at all the case for higher order corrections, including two gluon exchange, even though the infra-red sensitive region for the loop momentum of the quark pole contribution in the case of static screening coincides with that of Landau damping.

IV. CONCLUSION

In conclusion, we have derived the superconducting transition temperature with thermal diagrams in the normal phase. This ensures gauge invariance to all orders. We have also examined systematically the $O(g^4)$ corrections and found an additional contribution to the pre-exponential factor of T_c in the literature. Unlike the situations with static screening, Landau damping significantly improves the infrared behavior of the higher order diagrams and makes the perturbative expansion of the prefactor in terms of g legitimate.

In a forthcoming publication [16], we develop a perturbative formulation that enables us to eliminate the $O(1)$ uncertainty of the prefactor c of Eq. (1.1). Since the perturbative expansion is given in terms of g or $g\ln g$, the application to realistic situations is in fact rather limited. As was pointed

out in [9], even with asymptotic freedom, $g = 0.67$ at $\mu = 10^{10}$ MeV, and yet the energy scale probed at the BNL Relativistic Heavy Ion Collider (RHIC) is only a few hundred Mev. Nevertheless, it reveals some novel properties of superconductivity induced by a long range interaction which has not yet been fully examined in the literature.

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